INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 7, 697–718 (1987)

MAGNETOHYDRODYNAMIC FLOW IN A RECTANGULAR DUCT

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SUMMARY

The magnetohydrodynamic flow of an incompressible, viscous, electrically conducting fluid in a rectangular duct, with an external magnetic field applied transverse to the flow, has been investigated. One of the duct's boundaries which is perpendicular to the magnetic field is taken partly insulated, partly conducting. An analytical solution has been developed for the velocity field and magnetic field by reducing the problem to the solution of a Fredholm integral equation of the second kind, which has been solved numerically. Solutions have been obtained for Hartmann numbers M up to 100. All the infinite series obtained are transformed to infinite integrals first and then to finite integrals which contain modified Bessel functions of the second kind. In this way, the difficulties associated with the computation of infinite integrals with oscillating integrands and slowly converging infinite series, the convergence of which is further affected for large values of M, have been avoided. It is found that, as M increases, boundary layers are formed near the non-conducting boundaries and in the interface region, and a stagnant region is developed in front of the conducting boundary for velocity field. The maximm value of magnetic field takes place on the conducting part. These behaviours are shown on some graphs.

KEY WORDS MHD Flows Ducts Channels

INTRODUCTION

The study of flows of conducting fluids in ducts in the presence of transverse magnetic fields is important, owing to its practical applications in magnetohydrodynamic (MHD) generators, pumps, accelerators and flowmeters. Various forms of the problem with different combinations of conducting and non-conducting walls have been considered by Shercliff,[1] Chang and Lundgren,[2] Gold,[3] Hunt,[4] and others. Grinberg[5,6] has formulated the problem with perfectly conducting walls parallel to the applied field and non-conducting walls perpendicular to the field and attempted an exact analysis using a Green's function method, but his result is incomplete. Later Hunt and Stewartson[7] and Chiang and Lundgren[8] have used boundary layer methods to cast the same problem into the form of an integral equation. Recently Singh and Agarwal[9] followed Grinberg's solution procedure for the analytical part but they solved the resulting singular integral equation numerically since it could not be solved easily. Hunt and Williams[10] investigated the MHD flow between two parallel non-conducting planes. Wenger[11] presented a variational formulation that gave exact solutions for the velocity profile and electric potential distribution for a duct with mixed boundary conditions, but the analysis was of a very complicated nature. Recently, Wu[12] and Singh and Lal[13–15] have applied finite element methods for solving steady and unsteady MHD channel flow problems for different wall conductances.

In all the references cited above for MHD flow in ducts, each wall is either completely a conductor or completely an insulator, but not a mixture of two. When the wall is a mixture of

0271–2091/87/070697–22$11.00

Received November 1986

conducting and insulating portions the problem is much more difficult, and an exact solution is out of the question. The present paper deals with the MHD duct flow problem with mixed boundaries. So, we consider the flow of an incompressible, viscous, electrically conducting fluid in a rectangular duct with an external magnetic field applied transverse to the flow. One of the boundaries perpendicular to the magnetic field is taken to be partly insulating and partly perfectly conducting. The problem is solved analytically by reducing it to the solution of a Fredholm integral equation of the second kind, which has been solved numerically. Several valid approximations have been made for large Hartmann numbers in the calculations of the kernel and the right-hand side function of this integral equation. All the infinite series obtained in the solution are transformed to infinite integrals first and then to finite integrals which contain modified Bessel functions of the second kind. So, we have avoided the difficulties associated with the computation of slowly converging infinite series, the convergence of which is affected by large values of the Hartmann number.

BASIC EQUATIONS

The equations governing steady, laminar, fully developed flow of an incompressible, viscous, electrically conducting fluid in a rectangular duct, subjected to a constant and uniform imposed magnetic field, are well known and are discussed by Shercliff,[1] Dragoş[16] and others. On using a standard non-dimensional form the governing equations take the form

$$\nabla^2 V + M\frac{\partial B}{\partial x} = -1, \quad \text{in} \quad R, \tag{1}$$

$$\nabla^2 B + M\frac{\partial V}{\partial x} = 0, \quad \text{in} \quad R \tag{2}$$

where R denotes the section of the duct, $V(x, y)$, $B(x, y)$ are the velocity and induced magnetic field, and M is the Hartmann number. Here it is assumed that the applied magnetic field is parallel to the x-axis. $V(x, y)$ and $B(x, y)$ are in the z-direction, which is the axis of the duct, and the fluid is driven down the duct by means of a constant pressure gradient. The duct walls are at $x = 0$, $x = a$ and $y = \pm b/2$ (Figure 1). The walls at $x = a$, $y = \pm b/2$ are insulated completely but the wall at $x = 0$ is conducting for a length l starting from the origin symmetrically, and the rest of this wall is also insulated.

Accordingly, the boundary conditions for the equations (1) and (2) relating to the configuration of the problem in Figure 1 are as follows:

$$V(0, y) = V(a, y) = 0, \quad -\frac{b}{2} \leqslant y \leqslant \frac{b}{2}, \tag{3a}$$

$$V\left(x, \pm\frac{b}{2}\right) = 0, \quad 0 \leqslant x \leqslant a, \tag{3b}$$

$$B\left(x, \pm\frac{b}{2}\right) = 0, \quad 0 \leqslant x \leqslant a, \tag{3c}$$

$$B(a, y) = 0, \quad -\frac{b}{2} \leqslant y \leqslant \frac{b}{2}, \tag{3d}$$

$$B(0, y) = 0, \quad |y| > l, \tag{3e}$$

$$\frac{\partial B}{\partial x}(0, y) = 0, \quad 0 \leqslant |y| \leqslant l. \tag{3f}$$

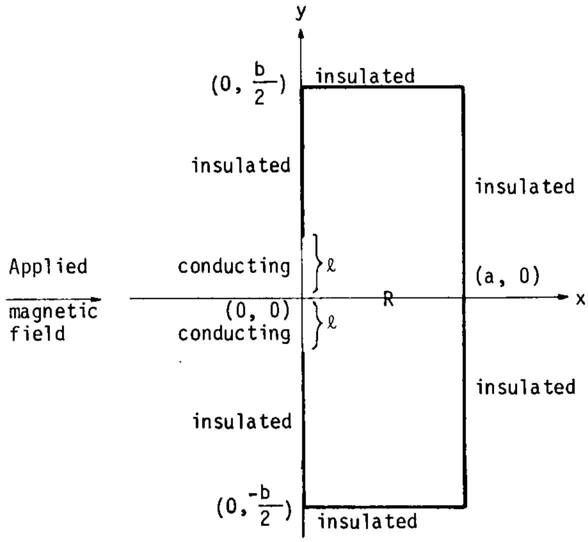

Figure 1.

ANALYTICAL SOLUTION

We split the solution into two parts as

$$\begin{pmatrix} V \\ B \end{pmatrix} = \begin{pmatrix} V_0 \\ B_0 \end{pmatrix} + \begin{pmatrix} V_1 \\ B_1 \end{pmatrix} \tag{4}$$

where '0' refers to the flow when the wall at $x = 0$ is insulated. We shall term it the primary flow. The solution corresponding to suffix '1' gives the correction due to the conducting part of the boundary and we shall designate the flow due to it as secondary flow.

Thus, we have

$$\nabla^2 V_0 + M \frac{\partial B_0}{\partial x} = -1, \tag{5}$$

$$\nabla^2 B_0 + M \frac{\partial V_0}{\partial x} = 0, \tag{6}$$

with the boundary conditions

$$V_0(0, y) = V_0(a, y) = 0, \quad -\frac{b}{2} \leqslant y \leqslant \frac{b}{2}, \tag{7a}$$

$$B_0(0, y) = B_0(a, y) = 0, \quad -\frac{b}{2} \leqslant y \leqslant \frac{b}{2}, \tag{7b}$$

$$V_0\left(x, \pm\frac{b}{2}\right) = 0, \quad 0 \leqslant x \leqslant a, \tag{7c}$$

$$B_0\left(x, \pm\frac{b}{2}\right) = 0, \quad 0 \leqslant x \leqslant a, \tag{7d}$$

and

$$\nabla^2 V_1 + M\frac{\partial B_1}{\partial x} = 0, \tag{8}$$

$$\nabla^2 B_1 + M\frac{\partial V_1}{\partial x} = 0, \tag{9}$$

with

$$V_1(0, y) = V_1(a, y) = 0, \quad -\frac{b}{2} \leqslant y \leqslant \frac{b}{2}, \tag{10a}$$

$$V_1\left(x, \pm\frac{b}{2}\right) = 0, \qquad 0 \leqslant x \leqslant a, \tag{10b}$$

$$B_1\left(x, \pm\frac{b}{2}\right) = 0, \qquad 0 \leqslant x \leqslant a, \tag{10c}$$

$$B_1(a, y) = 0, \qquad -\frac{b}{2} \leqslant y \leqslant \frac{b}{2}, \tag{10d}$$

$$B_1(0, y) = 0, \qquad l < |y|, < \frac{b}{2}, \tag{10e}$$

$$\frac{\partial B_1}{\partial x}(0, y) = -\frac{\partial B_0}{\partial x}(0, y), \qquad -l < y < l. \tag{10f}$$

In view of the symmetry about the x-axis, we need to consider the solution only in the region $(0 \leqslant x \leqslant a) \cap (0 \leqslant y \leqslant b/2)$.

The solution for the primary flow is due to Shercliff[1] and is given by

$$V_0(x, y) = \sum_{m=1,3}^{\infty} \frac{4b^2}{m^3\pi^3} \sin\left(\frac{m\pi}{2}\right)$$

$$\left\{1 + \frac{\mathrm{ch}\left(\frac{Mx}{2}\right)\mathrm{sh}[\rho_m(x-a)] - \mathrm{sh}(\rho_m x)\mathrm{ch}\left[\frac{M}{2}(x-a)\right]}{\mathrm{sh}\,\rho_m a}\right\}\cos\left(\frac{m\pi y}{b}\right), \tag{11}$$

$$B_0(x, y) = \sum_{m=1,3}^{\infty} \frac{4b^2}{m^3\pi^3} \sin\left(\frac{m\pi}{2}\right)$$

$$\left\{\frac{\mathrm{sh}(\rho_m x)\mathrm{sh}\left[\frac{M}{2}(x-a)\right] - \mathrm{sh}\left(\frac{Mx}{2}\right)\mathrm{sh}[\rho_m(x-a)]}{\mathrm{sh}\,\rho_m a}\right\}\cos\left(\frac{m\pi y}{b}\right), \tag{12}$$

where

$$\rho_m^2 = \frac{m^2\pi^2}{b^2} + \frac{M^2}{4},$$

and $\mathrm{sh}(x)$ and $\mathrm{ch}(x)$ are the sine hyperbolic and cosine hyperbolic functions, respectively.

To find the solution of the secondary flow, we expand V_1 and B_1 in Fourier cosine series as follows:

$$V_1(x, y) = \sum_{m=1,3}^{\infty} v_m(x)\cos\left(\frac{m\pi y}{b}\right), \tag{13}$$

$$B_1(x,y)=\sum_{m=1,3}^{\infty} b_m(x)\cos\left(\frac{m\pi y}{b}\right). \tag{14}$$

Substituting for V_1 and B_1 in equations (8) and (9) we obtain the system of ordinary differential equations

$$v_m''+Mb_m'-\frac{m^2\pi^2}{b^2}v_m=0,$$

$$b_m''+Mu_m'-\frac{m^2\pi^2}{b^2}b_m=0,$$

where the general solution is

$$v_m(x)=\mathrm{e}^{-(M/2)x}[(A_m\mathrm{ch}(\rho_m x)+B_m\mathrm{sh}(\rho_m x)]+\mathrm{e}^{(M/2)x}[C_m\mathrm{ch}(\rho_m x)+D_m\mathrm{sh}(\rho_m x)], \tag{15}$$

$$b_m(x)=\mathrm{e}^{-(M/2)x}[A_m\mathrm{ch}(\rho_m x)+B_m\mathrm{sh}(\rho_m x)]-\mathrm{e}^{(M/2)x}[C_m\mathrm{ch}(\rho_m x)+D_m\mathrm{sh}(\rho_m x)]. \tag{16}$$

The constants of integrations A_m, B_m, C_m, D_m are reduced to one set of unknowns A_m using the boundary conditions (10a)–(10d), and finally $V_1(x,y)$ and $B_1(x,y)$ are obtained through equations (13) and (14) as

$$V_1(x,y)=2\mathrm{sh}\left(\frac{Mx}{2}\right)\sum_{m=1,3}^{\infty}A_m\frac{\mathrm{sh}[\rho_m(x-a)]}{\mathrm{sh}(\rho_m a)}\cos\left(\frac{m\pi y}{b}\right), \tag{17}$$

$$B_1(x,y)=-2\mathrm{ch}\left(\frac{Mx}{2}\right)\sum_{m=1,3}^{\infty}A_m\frac{\mathrm{sh}[\rho_m(x-a)]}{\mathrm{sh}(\rho_m a)}\cos\left(\frac{m\pi y}{b}\right). \tag{18}$$

The set of unknowns A_m is to be determined by using the remaining boundary conditions (10e), (10f). So, substituting $B_1(x,y)$ in these boundary conditions, we obtain the following dual series equations for A_m:

$$\sum_{m=1,3}^{\infty}A_m\cos\left(\frac{m\pi y}{b}\right)=0,\quad l\leqslant y\leqslant\frac{b}{2}, \tag{19}$$

$$\sum_{m=1,3}^{\infty}A_m\rho_m\mathrm{cth}(\rho_m a)\cos\left(\frac{m\pi y}{b}\right)=\frac{2b^2}{\pi^3}\sum_{m=1,3}^{\infty}\frac{\sin\left(\frac{m\pi}{2}\right)}{m^3}\left(\frac{M}{2}-\rho_m\frac{\mathrm{sh}\left(\frac{M}{2}a\right)}{\mathrm{sh}(\rho_m a)}\right)$$
$$\cos\left(\frac{m\pi y}{b}\right),\quad 0\leqslant y<l, \tag{20}$$

where $\mathrm{cth}(x)$ is cotangent hyperbolic function.

In its simplest setting the set of equations (19)–(20) was solved by Sneddon[17] who obtained an exact solution. For more complicated equations, such as above, Keer and Sve[18] introduced a very effective integral representation which reduced the dual series equations to a Fredholm integral equation of the second kind. Following Keer and Sve,[18] we choose a representation for A_m as

$$A_m=\int_0^l f(t)J_0\left(\frac{m\pi t}{b}\right)\mathrm{d}t, \tag{21}$$

where $J_0(x)$ is the Bessel function of the first kind of order zero.

With this representation equation (19) is automatically satisfied on account of the identity[19,20]

$$\sum_{m=1,3}^{\infty} J_0(mt)\cos(mx) = \tfrac{1}{2}(t^2 - x^2)^{-1/2}H(t-x), \quad x+t<\pi, \tag{22}$$

where $H(x)$ is the Heaviside function. Integrating equation (20) with respect to y from 0 to y, and then substituting A_m from equation (21) with the help of the identity[19]

$$\sum_{m=1,3}^{\infty} J_0(mt)\sin(mx) = \tfrac{1}{2}(x^2 - t^2)^{-1/2}H(x-t) + \int_0^{\infty} \frac{I_0(st)\mathrm{sh}(sx)}{\mathrm{e}^{\pi s}+1}\mathrm{d}s, \quad x+t<\pi, \tag{23}$$

we obtain an Abel integral equation

$$\int_0^y \frac{f(t)}{\sqrt{(y^2-t^2)}}\mathrm{d}t = p(y), \tag{24}$$

where

$$\begin{aligned} p(y) = \frac{2\pi}{b}\Bigg\{ & \frac{2b^3}{\pi^4}\sum_{m=1,3}^{\infty} \frac{\sin\left(\frac{m\pi}{2}\right)}{m^4}\left[\frac{M}{2} - \frac{\mathrm{sh}\left(\frac{M}{2}a\right)}{\mathrm{sh}(\rho_m a)}\rho_m\right]\sin\left(\frac{m\pi y}{b}\right) \\ & - \int_0^l f(t)\sum_{m=1,3}^{\infty}\left[\frac{b}{\pi}\frac{\rho_m}{m}\mathrm{cth}(\rho_m a) - 1\right]J_0\left(\frac{m\pi t}{b}\right)\sin\left(\frac{m\pi y}{b}\right)\mathrm{d}t \\ & - \int_0^l f(t)\int_0^{\infty}\frac{I_0\left(\frac{\pi ts}{b}\right)\mathrm{sh}\left(\frac{\pi ys}{b}\right)}{1+\mathrm{e}^{\pi s}}\mathrm{d}s\mathrm{d}t\Bigg\}. \end{aligned} \tag{25}$$

Here $I_0(x)$ is the modified Bessel function of the first kind and of order zero.

The solution of Abel's integral equation (24) is given by Sneddon;[17]

$$f(t) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{yp(y)}{\sqrt{(t^2-y^2)}}\mathrm{d}y. \tag{26}$$

Substituting $p(y)$ in (26) and making use of some well known identities,[19] equation (26) is reduced to a Fredholm integral equation of the second kind for $f(t)$:

$$f(t) + \int_0^l K(x,t)f(x)\,\mathrm{d}x = g(t), \quad 0<t<l, \tag{27}$$

where the kernel $K(x,t)$ is

$$\begin{aligned} K(x,t) = \frac{2\pi^2}{b^2}t\Bigg\{ & \sum_{m=1,3}^{\infty} m\left[\frac{b}{\pi}\frac{\rho_m}{m}\mathrm{cth}(\rho_m a) - 1\right]J_0\left(\frac{m\pi x}{b}\right)J_0\left(\frac{m\pi t}{b}\right) \\ & + \int_0^{\infty}\frac{sI_0\left(\frac{\pi sx}{b}\right)I_0\left(\frac{\pi st}{b}\right)}{1+\mathrm{e}^{\pi s}}\mathrm{d}s\Bigg\} \end{aligned} \tag{28}$$

and the free term $g(t)$ is

$$g(t) = \frac{4b}{\pi^2}t\sum_{m=1,3}^{\infty}\frac{\sin\left(\frac{m\pi}{2}\right)}{m^3}\left[\frac{M}{2} - \frac{\mathrm{sh}\left(\frac{M}{2}a\right)}{\mathrm{sh}(\rho_m a)}\rho_m\right]J_0\left(\frac{m\pi t}{b}\right). \tag{29}$$

Since there is no analytical solution of equation (27) it must be solved numerically. To solve the Fredholm integral equation we need the value of the kernel and the quantity g, which contain some infinite series and infinite integrals. Our aim is to solve equation (27) for medium and large values of the Hartmann number, $10 \leqslant M \leqslant 100$, and this is from where most of the numerical difficulties arise. So there are numerous computational considerations which must be taken into account before attempting to solve equation (27) numerically.

COMPUTATIONAL CONSIDERATIONS

One of the troubles we may encounter in obtaining numerical solutions of the Fredholm integral equation (27) is the slow convergence of its kernel $K(x,t)$. The second one is the need for discretization with small step sizes for large values of M, since the kernel is of order M in that case. So, before we start the numerical solution of equation (27), the kernel $K(x,t)$ and the free term $g(t)$ should be transformed to much more computationally efficient forms.

Since we are concerned with the MHD flow at intermediate to high values of the Hartmann number ($10 \leqslant M \leqslant 100$) it is reasonable to approximate $\mathrm{cth}(\rho_m a)$ in (28) by 1. Then the infinite series in the kernel can be transformed into an infinite integral by using contour integration (Appendix I, equation (49)). So, the kernel (28) takes the form

$$K(x,t) = t\int_0^\infty \left[\sqrt{\left(s^2+\frac{M^2}{4}\right)} - s\right] J_0(sx)J_0(st)\,\mathrm{d}s + 2t\int_{M/2}^\infty \left[\sqrt{\left(s^2-\frac{M^2}{4}\right)} - s\right] \frac{I_0(sx)I_0(st)}{1+\mathrm{e}^{bs}}\,\mathrm{d}s. \tag{30}$$

By taking $t=\rho l$ and $x=tl$ the Fredholm integral equation (27) may be rewritten as

$$\theta(\rho) + \int_0^1 K_1(t,\rho)\theta(t)\,\mathrm{d}t = h(\rho), \quad 0<\rho<1, \tag{31}$$

where

$$\theta(\rho) = f(l\rho)/l\rho, \quad h(\rho) = g(l\rho)/l\rho \tag{32}$$

and

$$K_1(t,\rho) = l^2 t\left\{\int_0^\infty \left[\sqrt{\left(s^2+\frac{M^2}{4}\right)} - s\right] J_0(stl)J_0(s\rho l)\,\mathrm{d}s + 2\int_{M/2}^\infty \left[\sqrt{\left(s^2-\frac{M^2}{4}\right)} - s\right] \frac{I_0(s\rho l)I_0(stl)}{1+\mathrm{e}^{bs}}\,\mathrm{d}s\right\}, \tag{33}$$

$$h(\rho) = \frac{4b}{\pi^2}\sum_{m=1,3}^\infty \frac{\sin\left(\frac{m\pi}{2}\right)}{m^3}\left[\frac{M}{2} - \frac{\mathrm{sh}\left(\frac{M}{2}a\right)}{\mathrm{sh}(\rho_m a)}\rho_m\right] J_0\left(\frac{m\pi l\rho}{b}\right). \tag{34}$$

The first infinite integral in the kernel (33) was evaluated with the help of the identity[21]

$$J_0(\alpha t)J_0(\alpha x) = \frac{1}{\pi}\int_0^\pi J_0[\alpha(t^2+x^2-2tx\cos\theta)^{1/2}]\,\mathrm{d}\theta$$

and the identity obtained in Appendix II (equation (54)). Now, the kernel $K_1(t,\rho)$ takes a much more computationally efficient form

$$K_1(t,\rho) = 2l^2 t\left\{\frac{M^2}{16\pi}\int_0^\pi \left[I_1\left(\frac{M}{4}r\right)K_1\left(\frac{M}{4}r\right) + I_0\left(\frac{M}{4}r\right)K_0\left(\frac{M}{4}r\right)\right]\mathrm{d}\theta\right.$$

$$+\int_{M/2}^{\infty}\left[\sqrt{\left(s^2-\frac{M^2}{4}\right)}-s\right]\frac{I_0(sl\rho)I_0(slt)}{1+e^{bs}}\,ds\Bigg\},\tag{35}$$

where $r^2 = l^2(\rho^2 + t^2 - 2\rho t\cos\theta)$ and $I_0(x)$, $I_1(x)$ and $K_0(x)$, $K_1(x)$ are the modified Bessel functions of the first and second kinds of orders zero and one, respectively.

The second infinite integral above was evaluated numerically. The free term $h(\rho)$ in the integral equation (31), given by equation (34) can also be modified by summing the terms in $M/2$ analytically with the help of the following result (Appendix III):

$$\sum_{m=1,3}^{\infty}\frac{J_0(mt)\sin(mx)}{m^3}=\begin{cases}\dfrac{\pi^2 x}{8}-\dfrac{3}{8}x\sqrt{(t^2-x^2)}-\dfrac{1}{8}(t^2-2x^2)\arcsin\left(\dfrac{x}{t}\right), & x<t,\\[2ex] \dfrac{\pi^2 x}{8}-\dfrac{\pi}{16}(t^2+2x^2), & x>t,\end{cases}$$

where $x + t \leqslant \pi$.

So, $h(\rho)$ takes the form

$$h(\rho)=\frac{Mb\pi}{16}\left(1-2\frac{l^2\rho^2}{b^2}\right)-\frac{4b}{\pi^2}\sum_{m=1,3}^{\infty}\frac{\sin\left(\dfrac{m\pi}{2}\right)}{m^3}\,\mathrm{sh}\left(\frac{M}{2}a\right)\frac{\rho_m}{\mathrm{sh}(\rho_m a)}J_0\left(\frac{m\pi l\rho}{b}\right).\tag{36}$$

Now, the Fredholm integral equation (31) will be solved with the kernel (35) and the free term (36) which are in a more easily computable form. For the solution we replace the integrals by numerical quadratures based on Gauss's formula, and a system of algebraic equation is obtained for the unknown function θ (therefore f through (32)) in the representation A_m (equation (21)). By solving this system of equations for f one can determine $V_1(x, y)$ and $B_1(x, y)$. By virtue of the equation for A_m, the value of the function f can be substituted back into $V_1(x, y)$, $B_1(x, y)$ as

$$V_1(x,y)=2\,\mathrm{sh}\left(\frac{M}{2}x\right)\int_0^l f(t)\sum_{m=1,3}^{\infty}\frac{\mathrm{sh}[\rho_m(x-a)]}{\mathrm{sh}(\rho_m a)}J_0\left(\frac{m\pi t}{b}\right)\cos\left(\frac{m\pi y}{b}\right)dt,\tag{37}$$

$$B_1(x,y)=-2\,\mathrm{ch}\left(\frac{M}{2}x\right)\int_0^l f(t)\sum_{m=1,3}^{\infty}\frac{\mathrm{sh}[\rho_m(x-a)]}{\mathrm{sh}(\rho_m a)}J_0\left(\frac{m\pi t}{b}\right)\cos\left(\frac{m\pi y}{b}\right)dt.\tag{38}$$

Since the term $\mathrm{sh}[\rho_m(x-a)]/\mathrm{sh}(\rho_m a)$ can be approximated by $e^{-\rho_m(2a-x)} - e^{-\rho_m x}$ for large M, we need the following type of series for the calculations of V_1 and B_1:

$$\sum_{m=1,3}^{\infty} e^{-k\rho_m}J_0\left(\frac{m\pi t}{b}\right)\cos\left(\frac{m\pi y}{b}\right),\quad \text{where } k>0.$$

This series has been transformed into an infinite integral (Appendix IV) using contour integration and found to be

$$\sum_{m=1,3}^{\infty} e^{-k\rho_m}J_0\left(\frac{m\pi t}{b}\right)\cos\left(\frac{m\pi y}{b}\right)=\frac{b}{2\pi}\int_0^{\infty}e^{-k\sqrt{(s^2+M^2/4)}}J_0(ts)\cos(ys)\,ds$$
$$-\frac{b}{\pi}\int_{M/2}^{\infty}\sin\left[k\sqrt{\left(s^2-\frac{M^2}{4}\right)}\right]\frac{I_0(st)\,\mathrm{ch}(ys)}{1+e^{bs}}\,ds.$$

Both of these infinite integrals can be transformed to finite integrals containing modified Bessel functions only (Appendixes V and VI) where the term $\mathrm{ch}(ys)/(e^{bs}+1)$ in the second infinite integral was written as $\frac{1}{2}[e^{-(b-y)s} + e^{-(b+y)s} - e^{-(2b-y)s} - e^{-(2b+y)s} + e^{-(3b-y)s} + \ldots]$ and approximated by taking the first term $\frac{1}{2}e^{-(b-y)s}$ only, since the other terms were very small.

Finally, $V_1(x, y)$ and $B_1(x, y)$ take the forms

$$V_1(x,y) = \frac{bM}{2\pi^2}\operatorname{sh}\left(\frac{M}{2}x\right)g(x,y), \tag{39}$$

$$B_1(x,y) = -\frac{bM}{2\pi^2}\left(\frac{M}{2}x\right)g(x,y), \tag{40}$$

where

$$\begin{aligned} g(x,y) = \int_0^l f(t) \int_0^\pi \Bigg\{ &(2a-x)\frac{K_1\left(\frac{M}{2}\sqrt{[(t\cos\theta+y)^2+(2a-x)^2]}\right)}{\sqrt{[(t\cos\theta+y)^2+(2a-x)^2]}} \\ &- x\frac{K_1\left(\frac{M}{2}\sqrt{[(t\cos\theta+y)^2+x^2]}\right)}{\sqrt{[(t\cos\theta+y)^2+x^2]}} \\ &+ x\frac{K_1\left(\frac{M}{2}\sqrt{[(t\cos\theta+(b-y))^2+x^2]}\right)}{\sqrt{[(t\cos\theta+(b-y))^2+x^2]}} \\ &- (2a-x)\frac{K_1\left(\frac{M}{2}\sqrt{[(t\cos\theta+(b-y))^2+(2a-x)^2]}\right)}{\sqrt{[(t\cos\theta+(b-y))^2+(2a-x)^2]}} \Bigg\}\, d\theta\, dt. \end{aligned} \tag{41}$$

By adding the primary solution $V_0(x, y)$, $B_0(x, y)$ from equations (11) and (12) to the secondary solution $V_1(x, y)$, $B_1(x, y)$ obtained above one can find the velocity $V(x, y)$ and the induced magnetic field $B(x, y)$ for the duct problem.

The magnetic field on the conducting portion of the mixed boundary can be found directly from the substitution of A_m into $B_1(0, y)$, since $B_0(0, y)$ is zero. Simplifying, we obtain

$$B(0,y) = \frac{b}{\pi}\int_y^l \frac{f(t)}{\sqrt{(t^2-y^2)}}\,dt, \quad 0 \leqslant y < l \tag{42}$$

and, with $t = y \operatorname{ch}\theta$,

$$B(0,y) = \frac{b}{\pi}\int_0^{\operatorname{arc\,ch}(l/y)} f(y\operatorname{ch}\theta)\,d\theta. \tag{43}$$

For $y = 0$

$$B(0,0) = \frac{b}{\pi}\int_0^l \frac{f(t)}{t}\,dt. \tag{44}$$

The function f was interpolated with Gauss–Legendre abscissae at the points $y \operatorname{ch}\theta$, using Lagrange interpolation. The induced magnetic field was computed from (43) and (44) on the conducting boundary by using Gauss–Legendre integration.

The infinite integral, from $M/2$ to ∞, in the kernel (33) was first transformed to the form (by taking $s = M/2u$)

$$\int_{M/2}^{\infty}\left[\sqrt{\left(s^2-\frac{M^2}{4}\right)}-s\right]\frac{I_0(sl\rho)I_0(slt)}{1+e^{bs}}\,ds=\frac{M^2}{4}\int_0^1\sqrt{(1-u^2)}\,\frac{1}{u^3}\,\frac{I_0\left(l\rho\frac{M}{2u}\right)I_0\left(lt\frac{M}{2u}\right)}{1+e^{bM/2u}}\,du$$

$$-\frac{M^2}{4}\int_0^1\frac{1}{u^3}\,\frac{I_0\left(l\rho\frac{M}{2u}\right)I_0\left(lt\frac{M}{2u}\right)}{1+e^{bM/2u}}\,du. \quad (45)$$

The first integral was extended to $[-1,1]$ and then Gauss–Chebyshev quadrature was used; the second integral was evaluated with the 16-point Gauss–Legendre integration formula.

NUMERICAL RESULTS AND DISCUSSION

The duct $(0\leqslant x\leqslant a)\cap(-b/2\leqslant y\leqslant b/2)$ $(a=1, b=1)$ was divided into a mesh by taking mesh sizes 0·05 in each direction. For the velocity field near $y=0$, finer meshes were chosen to obtain desired accuracy in the results. Throughout the computations double precision was used except for solving the system of linear algebraic equations, as the matrix solver LEQT2F (IMSL library) was only available in single precision.

Equal velocity and induced magnetic field lines have been drawn for $10\leqslant M\leqslant 100$ and for several values of l by using the SURFACE II contour package on the Honeywell Multics machine at the University of Calgary, Canada. The non-smoothness of some curves is due to the linear interpolation used in that package. In Figures 2 and 3, equal velocity lines have been depicted for

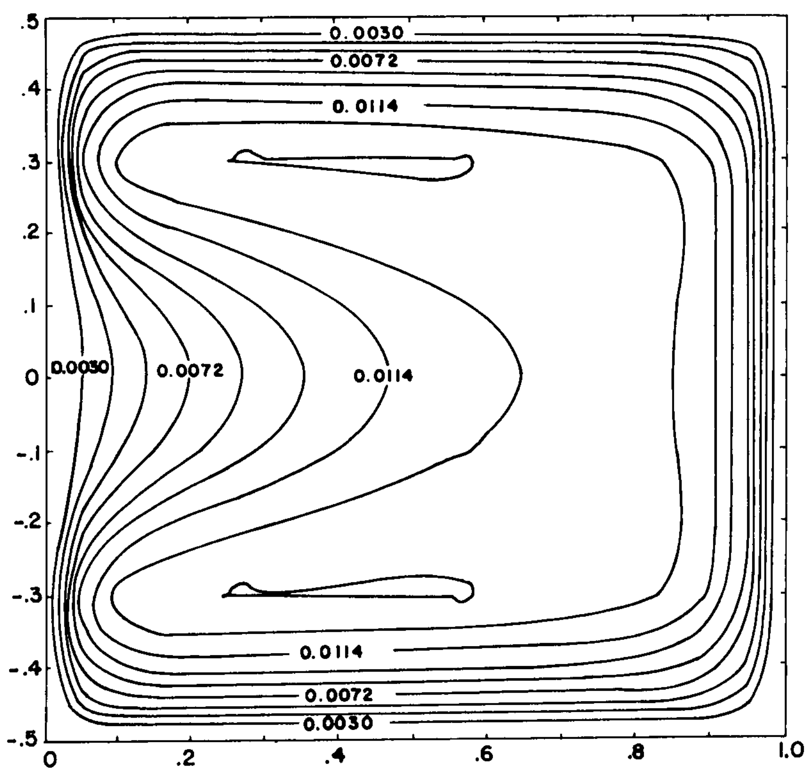

Figure 2. Velocity lines for $M=20, l=0{\cdot}25$

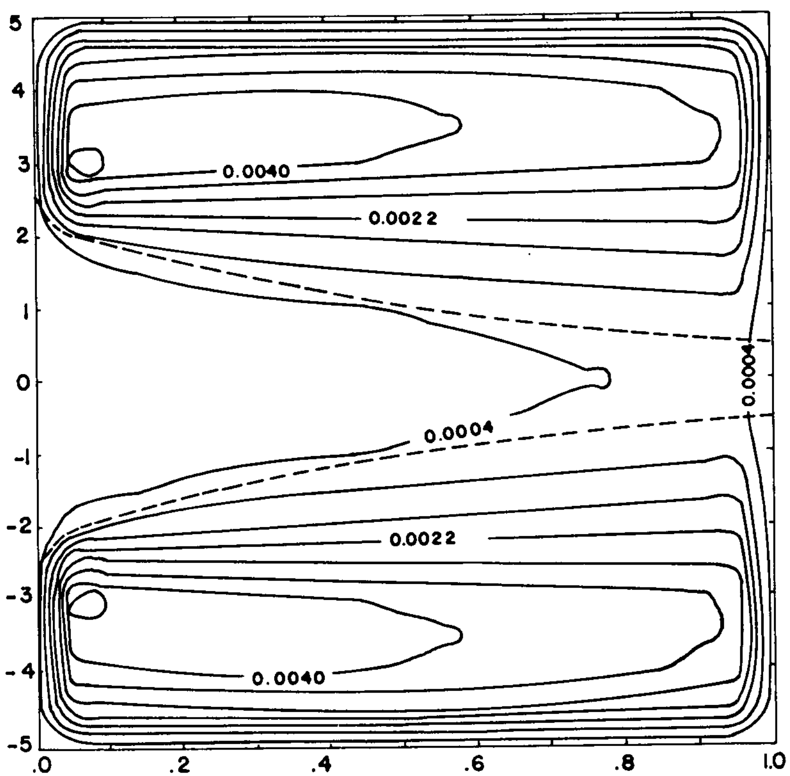

Figure 3. Velocity lines for $M = 100, l = 0{\cdot}25$

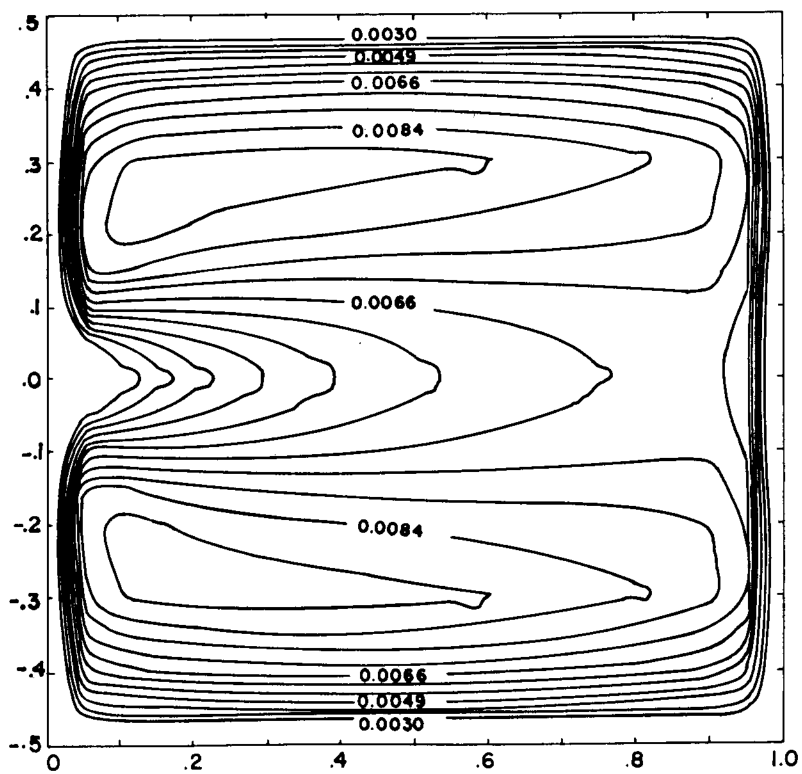

Figure 4. Velocity lines for $M = 50, l = 0{\cdot}1$

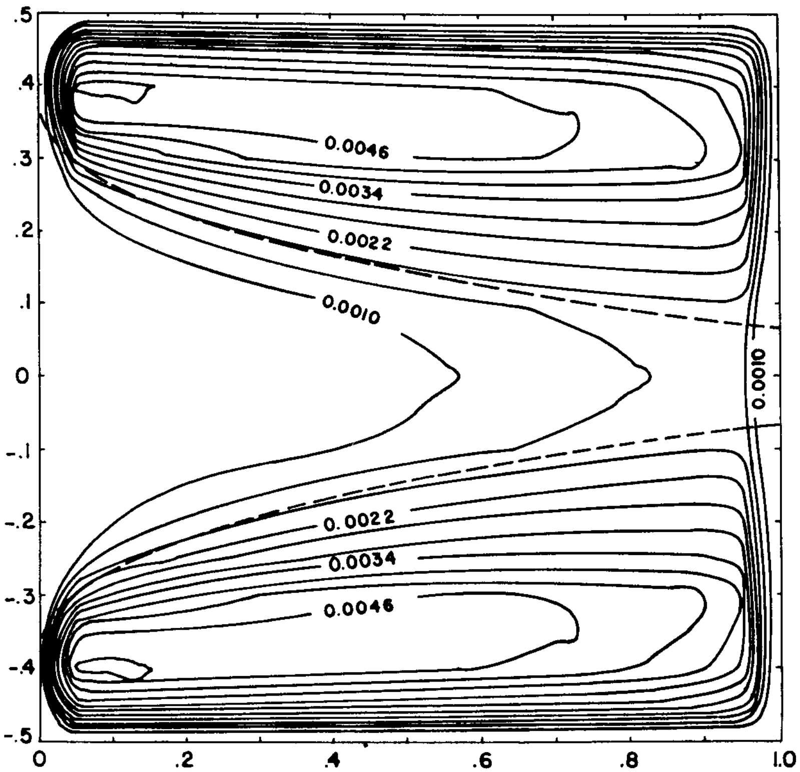

Figure 5. Velocity lines for $M = 50, l = 0{\cdot}35$

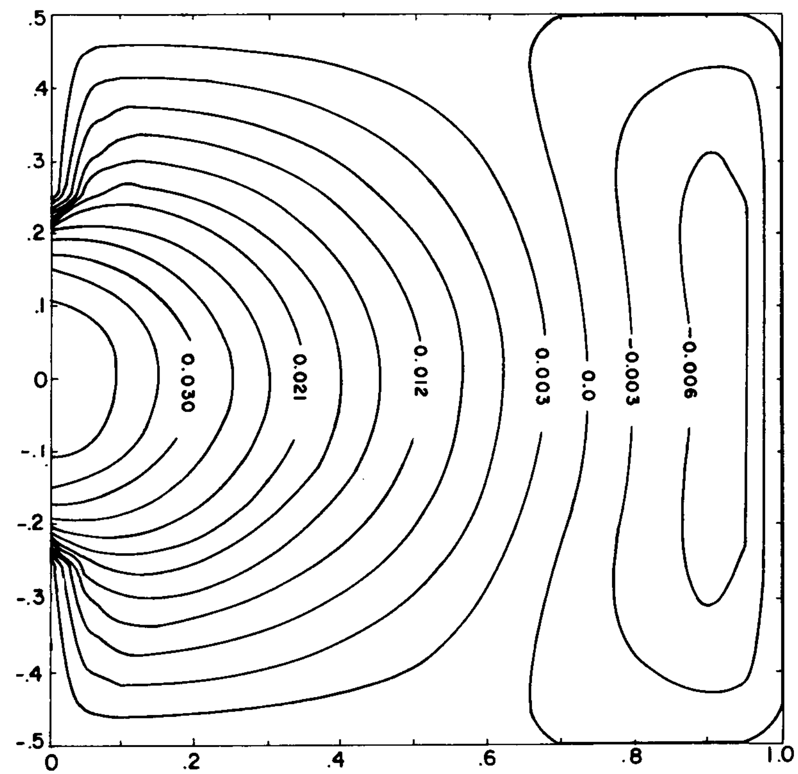

Figure 6. Magnetic field lines for $M = 20, l = 0{\cdot}25$

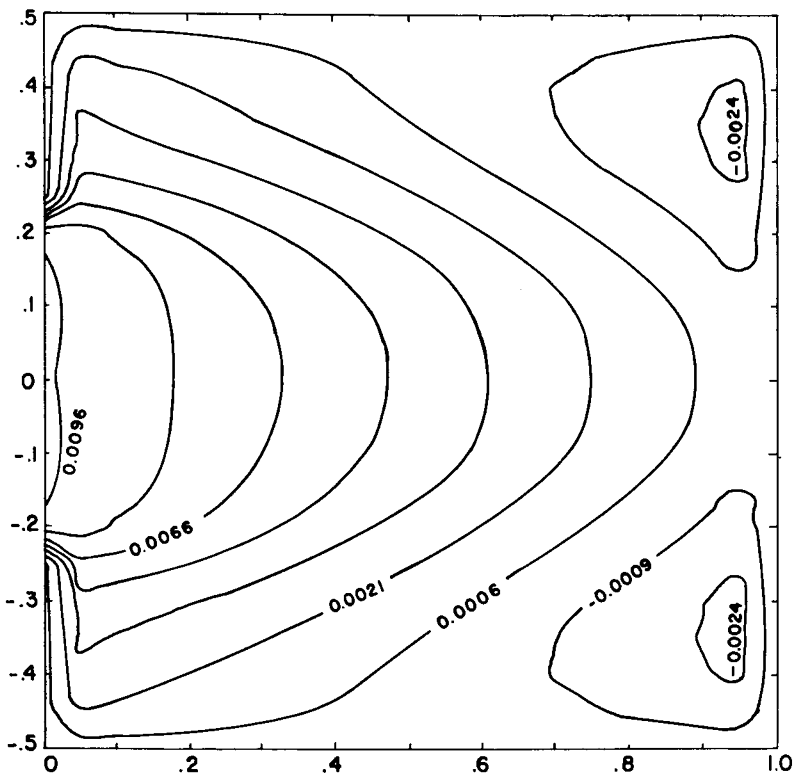

Figure 7. Magnetic field lines for $M = 100, l = 0{\cdot}25$

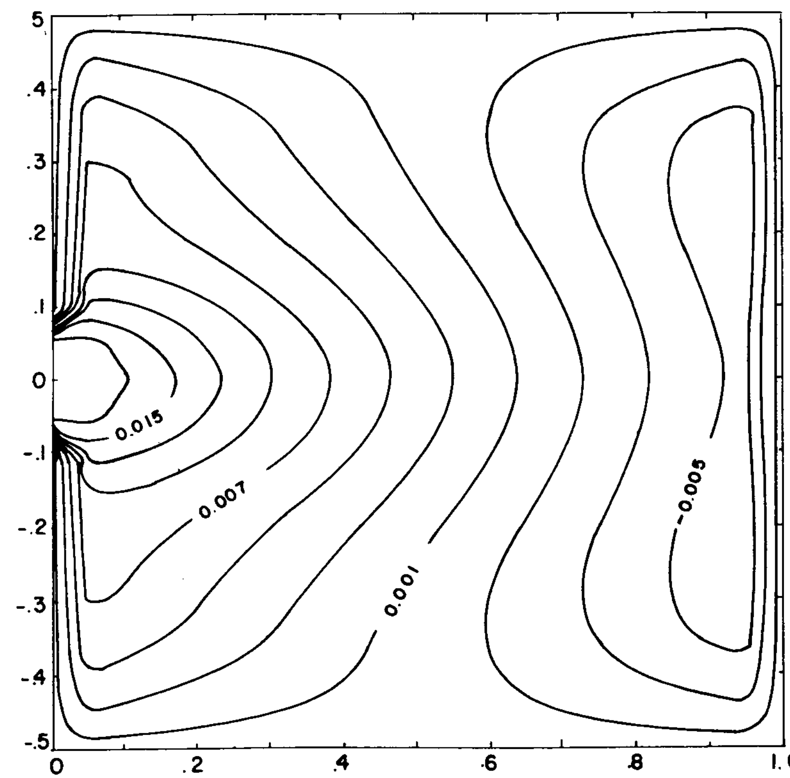

Figure 8. Magnetic field lines for $M = 50, l = 0{\cdot}1$

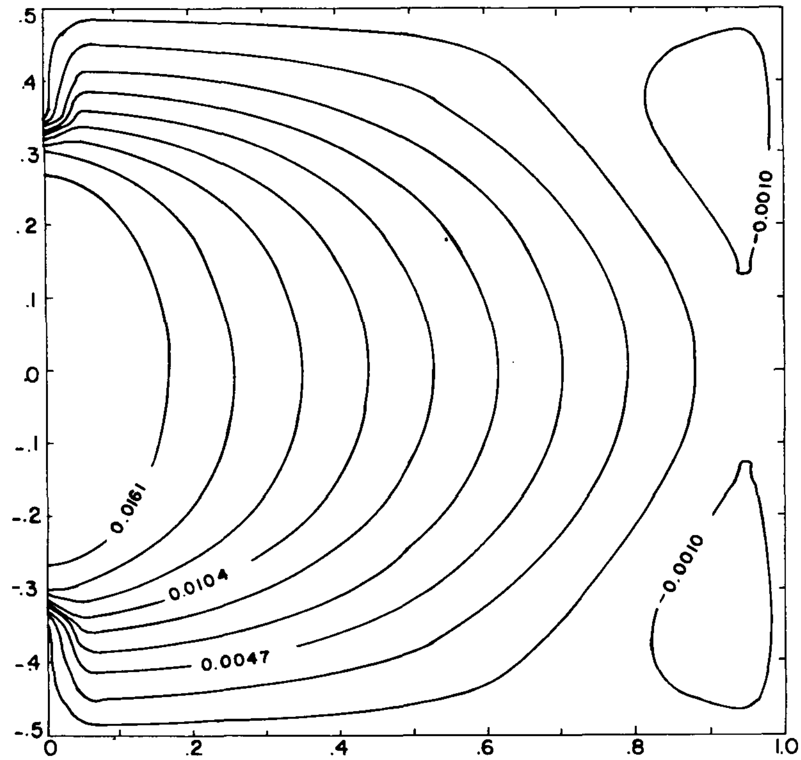

Figure 9. Magnetic field lines for $M = 50, l = 0{\cdot}35$

$l = 0{\cdot}25$ and for $M = 20$ and 100 respectively. As M increases, there is a formation of boundary layers of thickness M^{-1} on the non-conducting parts of the boundaries. The increase in the value of l results in enlargement of the stagnant region in front of the conducting portion, as is shown in Figures 4 and 5 for $M = 50$ and $l = 0{\cdot}1$, $l = 0{\cdot}35$, respectively. Also there is a parabolic boundary layer with thickness of order $M^{-1/2}$ at the points of discontinuity on the boundaries. This boundary layer has been shown using dotted lines; it is more pronounced for larger values of l or M. For smaller values of l or M the two layers have larger thicknesses and therefore tend to interfere with each other, so they are not demonstrated sufficiently.

The effect of varying M on the current lines (induced magnetic field lines) is shown in Figures 6 and 7 for $M = 20$ and 100, respectively. For the pattern of current lines the duct can be divided into two regions separated by the line $B = 0$. As we increase M, the region for which $B > 0$ keeps on increasing. Also the region characterized by $B < 0$ is split into two for M exceeding a certain critical value. The maximum value of B, of course, occurs on the conducting part.

The effect of increasing l, the conducting part, on current lines is similar to that of increasing M for a fixed l. This is shown in Figures 8 and 9, where $M = 50$.

The extension of the solution procedure used here to two rectangular ducts connected by a barrier which is partially a conductor and partially an insulator is given in Reference 22.

ACKNOWLEDGEMENT

Supported by The Scientific and Technical Research Council of Turkey.

APPENDIX I

Consider the integral of

$$F(z) = [\sqrt{(z^2+u^2)} - z] J_0(az) J_0(bz) \exp\left(\frac{i\pi z}{2}\right) \sec\left(\frac{\pi z}{2}\right) \tag{46}$$

taken along the contour shown in Figure 10.

We can show that

$$\int_\Gamma F(z)\,dz = 0, \quad \text{as} \quad R \to \infty$$

and

$$\int_\gamma F(z)\,dz = 0 \quad \text{as} \quad \rho \to 0.$$

Hence we have

$$\begin{aligned}
&\int_0^\infty [\sqrt{(t^2+u^2)} - t] J_0(at) J_0(bt) \exp\left(\frac{i\pi t}{2}\right) \sec\left(\frac{\pi t}{2}\right) dt \\
&\quad + \lim_{\rho\to 0} \sum_{m=1,3}^{\infty} \int_\pi^0 [\sqrt{(m^2+u^2)} - m] J_0(am) J_0(bm) \\
&\quad \exp\left[\frac{i\pi}{2}(m+\rho e^{i\theta})\right] \sec\left[\frac{\pi}{2}(m+\rho e^{i\theta})\right] i\rho e^{i\theta}\, d\theta \qquad (47) \\
&\quad + \int_\infty^{u_+} [\sqrt{(u^2-y^2)} - iy] J_0(iay) J_0(iby) \exp\left(-\frac{\pi}{2}y\right) \operatorname{sech}\left(\frac{\pi y}{2}\right) i\,dy \\
&\quad + \int_{u_-}^{0} [\sqrt{(u^2-y^2)} - iy] J_0(iay) J_0(iby) \exp\left(-\frac{\pi}{2}y\right) \operatorname{sech}\left(\frac{\pi y}{2}\right) i\,dy = 0.
\end{aligned}$$

By simplifying we obtain

$$\begin{aligned}
&\int_0^\infty [\sqrt{(t^2+u^2)} - t] J_0(at) J_0(bt) \left[1 + i\tan\left(\frac{\pi t}{2}\right)\right] dt \\
&\quad - 2 \sum_{m=1,3}^{\infty} [\sqrt{(m^2+u^2)} - m] J_0(am) J_0(bm) \\
&\quad - 2 \int_0^\infty [y - \sqrt{(y^2-u^2)}] \frac{I_0(ay) I_0(by)}{e^{\pi y}+1}\, dy
\end{aligned}$$

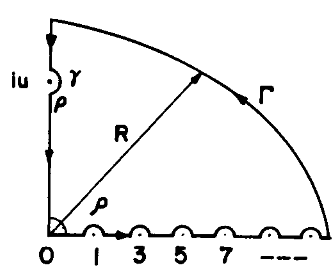

Figure 10.

$$-2\int_0^u [y+i\sqrt{(u^2-y^2)}]\frac{I_0(ay)I_0(by)}{e^{\pi y}+1}\,dy=0. \tag{48}$$

Taking real parts only we obtain

$$\sum_{m=1,3}^{\infty}[\sqrt{(m^2+u^2)}-m]J_0(am)J_0(bm)=\tfrac{1}{2}\int_0^\infty[\sqrt{(t^2+u^2)}-t]J_0(at)J_0(bt)\,dt$$
$$-\int_0^\infty\frac{yI_0(ay)I_0(by)}{e^{\pi y}+1}\,dy+\int_u^\infty\sqrt{(y^2-u^2)}\frac{I_0(ay)I_0(by)}{e^{\pi y}+1}\,dy. \tag{49}$$

APPENDIX II

Consider (Reference 20, p. 47)

$$J=\int_0^\infty(\beta^2+x^2)^{-1/2}e^{-\alpha(\beta^2+x^2)^{1/2}}J_0(\gamma x)\,dx=I_0(\beta p_-)K_0(\beta p_+), \tag{50}$$

where

$$p_-=\tfrac{1}{2}[(\alpha^2+\gamma^2)^{1/2}-\alpha],$$
$$p_+=\tfrac{1}{2}[(\alpha^2+\gamma^2)^{1/2}+\alpha].$$

So,

$$\frac{dp_-}{d\alpha}=-\frac{p_-}{\sqrt{(\alpha^2+\gamma^2)}},\quad \frac{dp_+}{d\alpha}=\frac{p_+}{\sqrt{(\alpha^2+\gamma^2)}}. \tag{51}$$

Taking the derivative of J with respect to α:

$$\frac{dJ}{d\alpha}=-\int_0^\infty e^{-\alpha(\beta^2+x^2)^{1/2}}J_0(\gamma x)\,dx=-\frac{\beta}{\sqrt{(\alpha^2+\gamma^2)}}[I_0(\beta p_-)p_+K_1(\beta p_+)$$
$$+K_0(\beta p_+)p_-I_1(\beta p_-)].$$

Taking the second derivative of J with respect to α:

$$\frac{d^2J}{d\alpha^2}=\int_0^\infty(\beta^2+x^2)^{1/2}e^{-\alpha(\beta^2+x^2)^{1/2}}J_0(\gamma x)\,dx=\frac{\alpha\beta}{(\alpha^2+\gamma^2)^{3/2}}[p_+I_0(\beta p_-)K_1(\beta p_+) \tag{52}$$
$$+p_-I_1(\beta p_-)K_0(\beta p_+)]+\frac{\beta^2}{\alpha^2+\gamma^2}[(p_+^2+p_-^2)I_0(\beta p_-)K_0(\beta p_+)$$
$$+2p_+p_-I_1(\beta p_-)K_1(\beta p_+)].$$

Letting $\beta\to 0$ in (52), we obtain

$$\int_0^\infty xe^{-\alpha x}J_0(\gamma x)\,dx=\frac{\alpha}{(\alpha^2+\gamma^2)^{3/2}}. \tag{53}$$

Then subtracting (53) from (52) and letting $\alpha\to 0$, we obtain

$$\int_0^\infty[(\beta^2+x^2)^{1/2}-x]J_0(\gamma x)\,dx=\tfrac{1}{2}\beta^2[I_0(\tfrac{1}{2}\beta\gamma)K_0(\tfrac{1}{2}\beta\gamma)+I_1(\tfrac{1}{2}\beta\gamma)K_1(\tfrac{1}{2}\beta\gamma)]. \tag{54}$$

APPENDIX III

For the evaluation of

$$\sum_{m=1,3}^{\infty} \frac{J_0(mt)\sin(mx)}{m^2}, \quad t+x<\pi, \tag{55}$$

we start with the identity[19]

$$\sum_{m=1,3}^{\infty} J_0(mt)\cos(mx) = \begin{cases} \frac{1}{2}(t^2-x^2)^{-1/2}, & x<t, \\ 0, & x>t, \end{cases} \quad x+t<\pi. \tag{56}$$

Integrate (56) with respect to x, then

$$\sum_{m=1,3}^{\infty} \frac{J_0(mt)\sin(mx)}{m} = \begin{cases} \frac{1}{2}\text{arcs in}\left(\frac{x}{t}\right), & x<t, \\ \frac{\pi}{4}, & x>t. \end{cases} \tag{57}$$

Integrating (57) with respect to x again we get

$$\sum_{m=1,3}^{\infty} \frac{J_0(mt)}{m^2} - \sum_{m=1,3}^{\infty} \frac{J_0(mt)\cos(mx)}{m^2} = \begin{cases} \frac{1}{2}x\,\text{arcs in}\left(\frac{x}{t}\right) + \frac{1}{2}(t^2+x^2)^{1/2} - \frac{1}{2}t, & x<t, \\ \frac{\pi x}{4} - \frac{1}{2}t, & x>t. \end{cases}$$

Since[19]

$$\sum_{m=1,3}^{\infty} \frac{J_0(mt)}{m^2} = \frac{\pi^2}{8} - \frac{t}{2},$$

$$\sum_{m=1,3}^{\infty} \frac{J_0(mt)\cos(mx)}{m^2} = \begin{cases} \frac{\pi^2}{8} - \frac{1}{2}x\,\text{arcs in}\left(\frac{x}{t}\right) - \frac{1}{2}(t^2-x^2)^{1/2}, & x<t, \\ \frac{\pi^2}{8} - \frac{\pi x}{4}, & x>t. \end{cases} \tag{58}$$

Integrating (58) with respect to x, we obtain

$$\sum_{m=1,3}^{\infty} \frac{J_0(mt)\sin(mx)}{m^3} = \begin{cases} \frac{\pi^2 x}{8} - \frac{3}{8}x(t^2-x^2)^{1/2} - \frac{1}{8}(t^2+2x^2)\,\text{arcs in}\left(\frac{x}{t}\right), & x<t, \\ \frac{\pi^2 x}{8} - \frac{\pi}{16}(t^2+2x^2). & x>t, \end{cases} \quad x+t<\pi. \tag{59}$$

If $x=\pi/2$ and $t<\pi/2$ then

$$\sum_{m=1,3}^{\infty} \frac{J_0(mt)\sin\left(\frac{m\pi}{2}\right)}{m^3} = \frac{\pi}{32}(\pi^2-2t^2). \tag{60}$$

APPENDIX IV

Consider the integral of

$$F(z) = e^{-k\sqrt{(z^2+u^2)}} J_0(az)\cos(bz)\exp\left(\frac{i\pi z}{2}\right)\sec\left(\frac{\pi z}{2}\right), \tag{61}$$

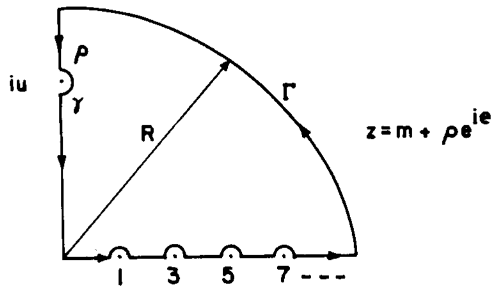

Figure 11.

taken along the contour shown in Figure 11.

We can show that

$$\int_{\Gamma} F(z)\mathrm{d}z = 0 \quad \text{as} \quad R \to \infty$$

and

$$\int_{\gamma} F(z)\mathrm{d}z = 0 \quad \text{as} \quad \rho \to 0.$$

Hence we have

$$\begin{aligned}&\int_0^\infty \mathrm{e}^{-k\sqrt{(x^2+u^2)}}\mathrm{e}^{\mathrm{i}\pi x/2}\sec\left(\frac{\pi x}{2}\right)J_0(ax)\cos(bx)\,\mathrm{d}x\\&+\lim_{\rho\to 0}\int_\pi^0\sum_{m=1,3}^{\infty}\mathrm{e}^{-k\sqrt{(m^2+u^2)}}J_0(am)\cos(bm)\mathrm{e}^{\mathrm{i}(\pi/2)(m+\rho\mathrm{e}^{\mathrm{i}\theta})}\sec\left[\frac{\pi}{2}(m+\rho\mathrm{e}^{\mathrm{i}\theta})\right]\mathrm{i}\rho\mathrm{e}^{\mathrm{i}\theta}\,\mathrm{d}\theta \qquad (62)\\&+\int_\infty^u \mathrm{e}^{-k\sqrt{(-y^2+u^2)}}J_0(\mathrm{i}\,ay)\cos(\mathrm{i}\,by)\mathrm{i}\,\mathrm{e}^{-\pi y/2}\operatorname{sech}\left(\frac{\pi y}{2}\right)\mathrm{d}y\\&+\int_u^0 \mathrm{e}^{-k\sqrt{(u^2-y^2)}}J_0(\mathrm{i}\,ay)\cos(\mathrm{i}\,by)\mathrm{i}\,\mathrm{e}^{-\pi y/2}\operatorname{sech}\left(\frac{\pi y}{2}\right)\mathrm{d}y = 0.\end{aligned}$$

Taking real parts only we obtain

$$\begin{aligned}&\sum_{m=1,3}^{\infty}\mathrm{e}^{-k\sqrt{(m^2+u^2)}}J_0(am)\cos(bm)\\&=\frac{1}{2}\int_0^\infty \mathrm{e}^{-k\sqrt{(x^2+u^2)}}J_0(ax)\cos(bx)\,\mathrm{d}x\\&\quad-\int_u^\infty \sin[k\sqrt{(y^2-u^2)}]\frac{I_0(ay)\mathrm{ch}(by)}{1+\mathrm{e}^{\pi y}}\mathrm{d}y, \qquad (63)\end{aligned}$$

since

$$\lim_{\rho\to 0}\frac{\mathrm{e}^{\mathrm{i}\pi m/2}\,\mathrm{e}^{\mathrm{i}\pi/2}\,\rho\mathrm{e}^{\mathrm{i}\theta}}{\cos\left(\frac{m\pi}{2}+\frac{\pi\rho}{2}\mathrm{e}^{\mathrm{i}\theta}\right)}\mathrm{i}\,\rho\mathrm{e}^{\mathrm{i}\theta} = \frac{2}{\pi}$$

and

$$\mathrm{Re}\left[\mathrm{i}e^{-k\sqrt{(y^2-u^2)}}I_0(ay)\,\mathrm{ch}\,(by)\,e^{-\pi y/2}\,\mathrm{sech}\,\frac{\pi}{2}\right]$$

$$=2\sin\left[k\sqrt{(y^2-u^2)}\right]I_0(ay)\,\mathrm{ch}(by)\frac{1}{1+e^{\pi y}}.$$

APPENDIX V

Consider the integral

$$R=\int_0^\infty e^{-k\sqrt{(u^2+\alpha^2)}}\cos(y\alpha)J_0(\alpha t)\mathrm{d}\alpha. \tag{64}$$

Since

$$J_0(z)=\frac{1}{\pi}\int_0^\pi\cos(z\cos\theta)\mathrm{d}\theta,$$

integral (64) can be written as

$$\begin{aligned}R&=\frac{1}{\pi}\int_0^\infty e^{-k\sqrt{(u^2+\alpha^2)}}\int_0^\pi\cos(\alpha t\cos\theta)\cos(y\alpha)\mathrm{d}\theta\mathrm{d}\alpha\\&=\frac{1}{2\pi}\int_0^\infty e^{-k\sqrt{u^2+\alpha^2}}\int_0^\pi[\cos\alpha(t\cos\theta+y)+\cos\alpha(t\cos\theta-y)]\mathrm{d}\theta\mathrm{d}\alpha\\&=\frac{1}{2\pi}\int_0^\pi\left[\int_0^\infty e^{-k\sqrt{(u^2+\alpha^2)}}\cos\alpha(t\cos\theta+y)\,\mathrm{d}\alpha\right.\\&\quad\left.+\int_0^\infty e^{-k\sqrt{(u^2+\alpha^2)}}\cos\alpha(t\cos\theta-y)\mathrm{d}\alpha\right]\mathrm{d}\theta.\end{aligned} \tag{65}$$

For the evaluation of the infinite integrals above we make use of the identity[19]

$$\int_0^\infty e^{-\beta\sqrt{(\gamma^2+x^2)}}\cos(ax)\frac{\mathrm{d}x}{\sqrt{(\gamma^2+x^2)}}=K_0[\gamma\sqrt{(\alpha^2+\beta^2)}]. \tag{66}$$

By taking the derivative of (66) with respect to β we arrive at

$$\int_0^\infty e^{-\beta\sqrt{(\gamma^2+x^2)}}\cos(ax)\mathrm{d}x=K_1[\sqrt{(\alpha^2+\beta^2)}]\frac{\gamma\beta}{\sqrt{(\alpha^2+\beta^2)}}. \tag{67}$$

Now, substituting (67) back into (65) we obtain

$$\begin{aligned}R&=\frac{uk}{2\pi}\int_0^\pi\left[\frac{K_1(u\sqrt{[(t\cos\theta+y)^2+k^2]})}{\sqrt{[(t\cos\theta+y)^2+k^2]}}+\frac{K_1(u\sqrt{[(t\cos\theta-y)^2+k^2]})}{\sqrt{[(t\cos\theta-y)^2+k^2]}}\right]\mathrm{d}\theta\\&=\frac{uk}{\pi}\int_0^\pi\frac{K_1(u\sqrt{[(t\cos\theta+y)^2+k^2]})}{\sqrt{([(t\cos\theta+y)^2+k^2])}}\mathrm{d}\theta.\end{aligned}$$

Therefore,

$$\int_0^\infty e^{-k\sqrt{u^2+\alpha^2}} \cos(y\alpha) J_0(at)\,d\alpha = \frac{uk}{\pi}\int_0^\pi \frac{K_1(u\sqrt{[(t\cos\theta+y)^2+k^2]})}{\sqrt{[(t\cos\theta+y)^2+k^2]}}\,d\theta. \tag{68}$$

APPENDIX VI

For the evaluation of the integral

$$\int_{M/2}^\infty \sin\left[k\sqrt{\left(s^2-\frac{M^2}{4}\right)}\right]\frac{I_0(ts)\mathrm{ch}(ys)}{1+e^{bs}}\,ds \tag{69}$$

We take $s^2-(M^2/4)=u^2$; then

$$\int_{M/2}^\infty \sin\left[k\sqrt{\left(s^2-\frac{M^2}{4}\right)}\right]\frac{I_0(ts)\mathrm{ch}(ys)}{1+e^{bs}}\,ds$$

$$=\int_0^\infty \sin(ku)\frac{I_0\left[t\sqrt{\left(u^2+\frac{M^2}{4}\right)}\right]\mathrm{ch}\left[y\sqrt{\left(u^2+\frac{M^2}{4}\right)}\right]}{1+e^{b\sqrt{(u^2+M^2/4)}}}\frac{u\,du}{\sqrt{\left(u^2+\frac{M^2}{4}\right)}}. \tag{70}$$

So, we need the integral of the following type:

$$\int_0^\infty e^{-\beta\sqrt{(u^2+p^2)}} I_0[t\sqrt{(u^2+p^2)}]\frac{u\sin(ku)}{\sqrt{(u^2+p^2)}}\,du. \tag{71}$$

For the evaluation of the above integral we consider the integral of

$$F(z)=e^{-\beta\sqrt{(z^2+p^2)}} I_0[t\sqrt{(z^2+p^2)}]\frac{e^{izk}}{\sqrt{(z^2+p^2)}},$$

taken along the contour shown in Figure 12.

We can show that $\int_C F(z)\,dz=0$ and

$$\int_\Gamma F(z)\,dz=0 \quad \text{as} \quad R\to\infty \quad \text{and} \quad \lim_{\rho\to 0}\int_\gamma F(z)\,dz=0.$$

Hence

$$\int_0^\infty e^{-\beta\sqrt{(u^2+p^2)}} I_0[t\sqrt{(u^2+p^2)}]\frac{e^{iku}}{\sqrt{(u^2+p^2)}}\,du$$

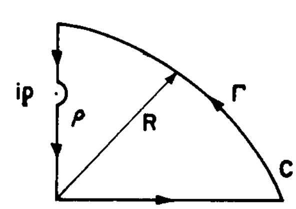

Figure 12.

$$+\int_{+\mathrm{i}\infty}^{\mathrm{i}(p_+)} F(\mathrm{i}v)(\mathrm{i}\mathrm{d}v) + \int_{\mathrm{i}(p_-)}^{0} F(\mathrm{i}v)\mathrm{i}\,\mathrm{d}v = 0. \tag{72}$$

Now,

$$\int_{+\mathrm{i}\infty}^{\mathrm{i}p_+} F(\mathrm{i}v)(\mathrm{i}\mathrm{d}v) = -\int_{p}^{\infty} \mathrm{e}^{-\mathrm{i}\beta\sqrt{(v^2-p^2)}} J_0[t\sqrt{(v^2-p^2)}]\frac{\mathrm{e}^{-kv}}{\sqrt{(v^2-p^2)}}\mathrm{d}v,$$

$$\int_{-\mathrm{i}p_-}^{0} F(\mathrm{i}v)(\mathrm{i}\mathrm{d}v) = -\mathrm{i}\int_{0}^{p} \mathrm{e}^{-\beta\sqrt{(p^2-v^2)}} I_0[t\sqrt{(p^2-v^2)}]\frac{\mathrm{e}^{-kv}}{\sqrt{(p^2-v^2)}}\mathrm{d}v.$$

Substituting back in (72)

$$\begin{aligned}&\int_0^{\infty} \mathrm{e}^{-\beta\sqrt{(u^2+p^2)}} I_0[t\sqrt{(u^2+p^2)}]\frac{\mathrm{e}^{\mathrm{i}ku}}{\sqrt{(u^2+p^2)}}\\ &\quad -\int_p^{\infty} \mathrm{e}^{-\mathrm{i}\beta\sqrt{(v^2-p^2)}} J_0[t\sqrt{(v^2-p^2)}]\frac{\mathrm{e}^{-kv}}{\sqrt{(v^2-p^2)}}\mathrm{d}v\\ &\quad -\mathrm{i}\int_0^{p} \mathrm{e}^{-\beta\sqrt{(p^2-v^2)}} I_0[t\sqrt{(p^2-v^2)}]\frac{\mathrm{e}^{-kv}}{\sqrt{(p^2-v^2)}}\mathrm{d}v = 0.\end{aligned} \tag{73}$$

Taking the real part we obtain

$$\begin{aligned}&\int_0^{\infty} \mathrm{e}^{-\beta\sqrt{(u^2+p^2)}} I_0[t\sqrt{(u^2+p^2)}]\frac{\cos(ku)}{\sqrt{(u^2+p^2)}}\mathrm{d}u\\ &\quad = \int_0^{\infty} \cos(\beta u) J_0(ut)\mathrm{e}^{-k\sqrt{(u^2+p^2)}}\frac{\mathrm{d}u}{\sqrt{(u^2+p^2)}}.\end{aligned} \tag{74}$$

Differentiating with respect to k, we obtain

$$\begin{aligned}\int_0^{\infty} \mathrm{e}^{-\beta\sqrt{(u^2+p^2)}} I_0[t\sqrt{(u^2+p^2)}]\frac{u\sin(ku)}{\sqrt{(u^2+p^2)}}\mathrm{d}u &= \int_0^{\infty}\cos(\beta u)J_0(ut)\mathrm{e}^{-k\sqrt{(u^2+p^2)}}\mathrm{d}u\\ &= \frac{pk}{\pi}\int_0^{\pi}\frac{K_1(p\sqrt{[(t\cos\theta+\beta)^2+K^2]})}{\sqrt{[(t\cos\theta+\beta)^2+k^2]}}\end{aligned} \tag{75}$$

(from Appendix V).

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